[jh240051] Analysis of Mathematical Structure of Reservoir Computing Model

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1 Introduction.

Reservoir computing, a brain-inspired machine-learning technique that employs a data-driven dynamical system, is effective in predicting time series and frequency spectra in chaotic behaviors, including fluid flow and global atmospheric dynamics [1, 2, 4, 5, 6].

The extent to which a data-driven model using reservoir computing can capture the dynamical properties of original systems should be determined. Lu *et al.* [1] reported that a data-driven model has an attractor similar to that of the original system under an appropriate choice of parameters. Nakai and Saiki [3] confirmed that a single data-driven model could infer the time series of chaotic fluid flow from various initial conditions. They suggested that a data-driven model could reconstruct the attractor of the original dynamical system.

In our project, we analyze a data-driven model constructed using reservoir computing from a dynamical system point of view. In particular, we focus on reconstructing the geometric structure. We investigate that the Lyapunov spectrum of the actual dynamical system underlying the training data, including the negative exponents, is reproduced within a relatively small subspace in the machine learning model obtained through reservoir computing.

2 Reservoir computation.

What's Reservoir computation?

- a relatively high-dimentional fixed neural-network composed of simple nonlinear dynamical systems
- determination of output layer
- For Lorenz system and Kuramoto–Sivashinsky system, inference [1, 5, 6]

2.1 Procedure of training.

◯ **1st step (making a reservoir vector)**

Making a reservoir vector $\mathbf{r}(t)$ correspond to decomposition of the input data $\mathbf{u}$ by using nonlinear function tanh:
$\mathbf{r}(t+\Delta t) = (1-\alpha)\mathbf{r}(t) + \alpha \tanh(\mathbf{A}\mathbf{r}(t) + \mathbf{W}_{\text{in}}\mathbf{u}(t))$.
$\mathbf{A}, \mathbf{W}_{\text{in}}$: sparse random matrix, whose maximal eigenvalue is controlled.

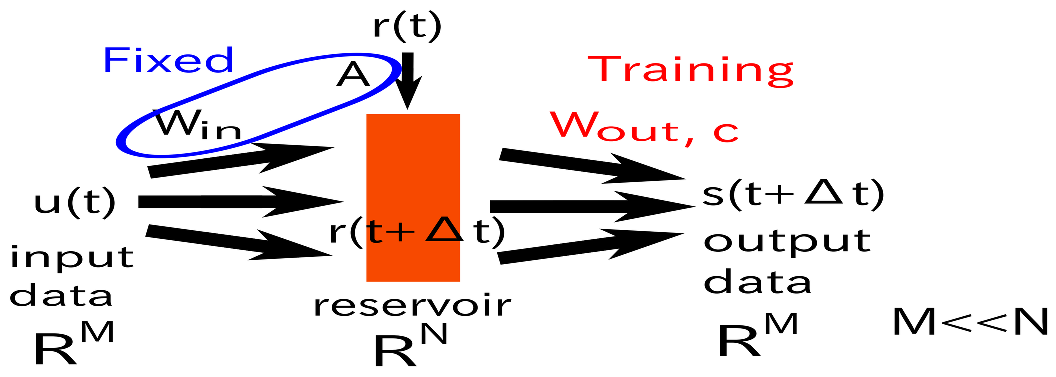

Figure 1: Schematic picture of a reservoir computing (training phase)

◯ **2nd step (determination of output layer)**

We determine $\mathbf{W}_{\text{out}}$ s.t.

$$t^{\forall} < T \qquad \mathbf{W}_{\text{out}}\mathbf{r}(t+\Delta t) \approx \mathbf{s}(t+\Delta t).$$

⇒ Determine them s.t. the form is minimized:

$$\sum_{l=1}^{L} \|(\mathbf{W}_{\text{out}}\mathbf{r}(l\Delta t)) - \mathbf{s}(l\Delta t)\|^2 + \beta[Tr(\mathbf{W}_{\text{out}}\mathbf{W}_{\text{out}}^T)].$$

2.2 Procedure of inference.

Using the $\mathbf{W}^*_{\text{out}}$, we infer the time-series $\mathbf{s}$.

$\hat{\mathbf{s}}(t) = \mathbf{W}^*_{\text{out}}\mathbf{r}(t)$
$\mathbf{r}(t+\Delta t) = (1-\alpha)\mathbf{r}(t) + \alpha \tanh(\mathbf{A}\mathbf{r}(t) + \mathbf{W}_{\text{in}}\hat{\mathbf{s}}(t))$.

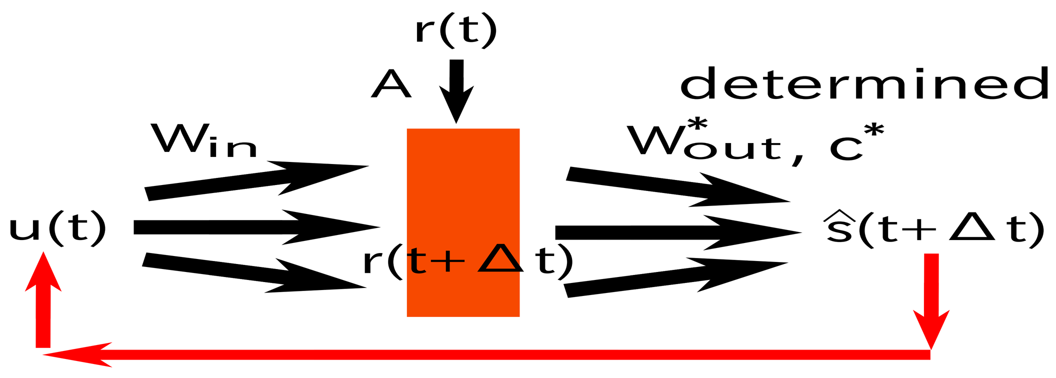

Figure 2: Schematic picture of a reservoir computing (prediction phase)

This reservoir system corresponds to the data-driven model of $\mathbf{u}$.

3 Our project.

The first example is the data-driven model of the Lorenz [7], a dissipative N-dimensional ODE which is a model of some oscillating scalar atmospheric quantity described as

$$\frac{dx_k}{dt} = x_{k-1}(x_{k+1} - x_{k-2}) - x_k + f, \text{ for } k = 1, \ldots, N, \quad (1)$$

where the system has cyclical symmetry so $x_{N+k} = x_k$ for all $k = 1, \ldots, N$, and where f is a forcing parameter.

We analyze machine learning models obtained through reservoir computing using examples such as high-dimensional Lorenz systems, which are often used in the field of meteorology. There are some parameters that need to be artificially set in machine learning (so-called hyperparameters), and we aim to clarify the reproducibility of these settings and the structures.

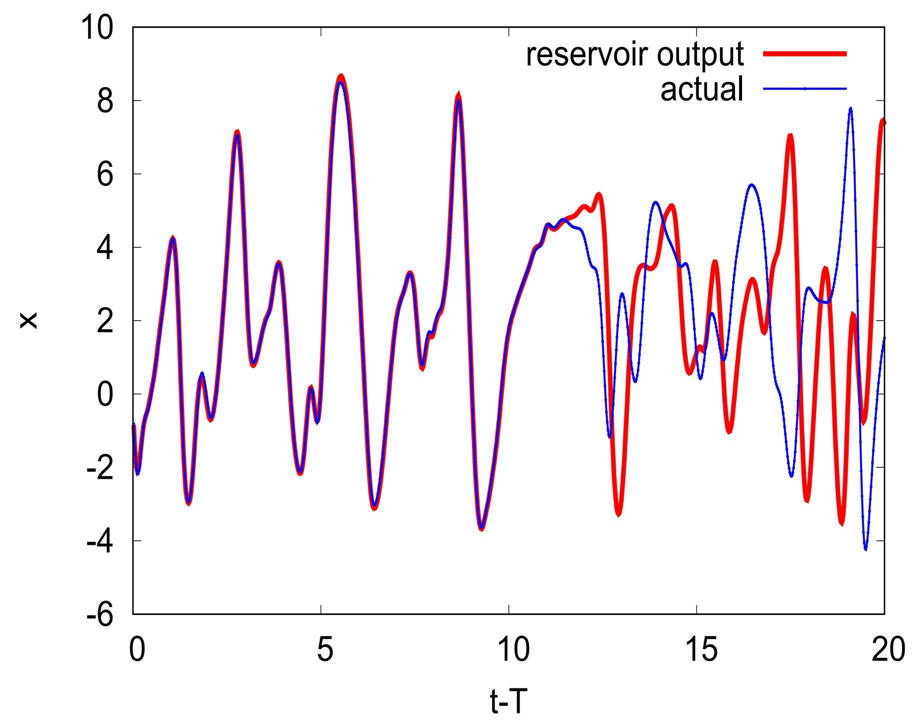

Figure 3: Short time inference of a variable of the Lorenz-96 system

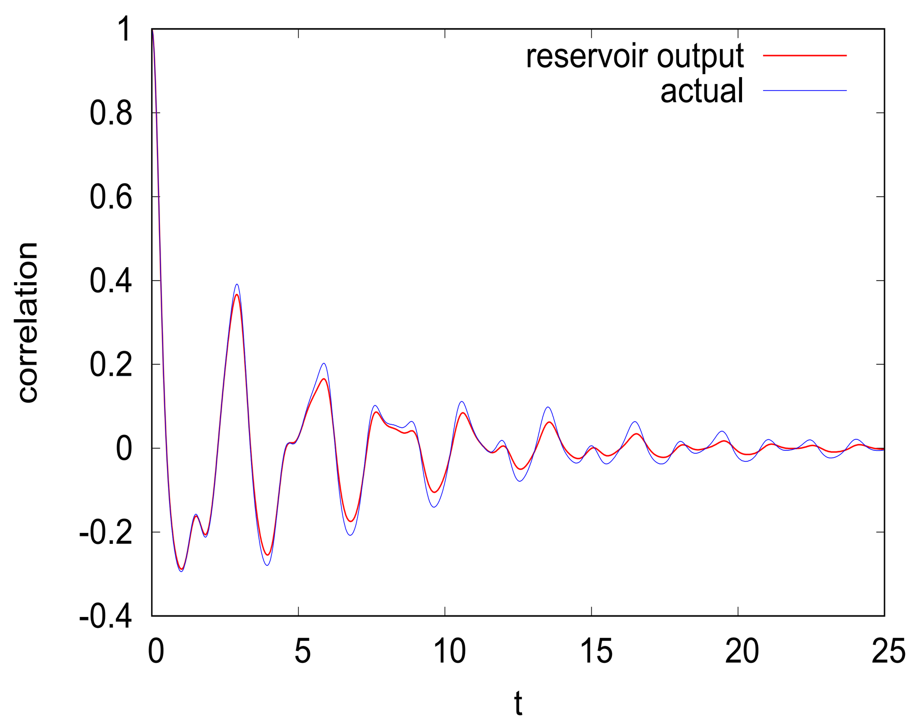

Figure 4: Autocorrelation functions for the data-driven model and the actual system of the Lorenz-96 system.

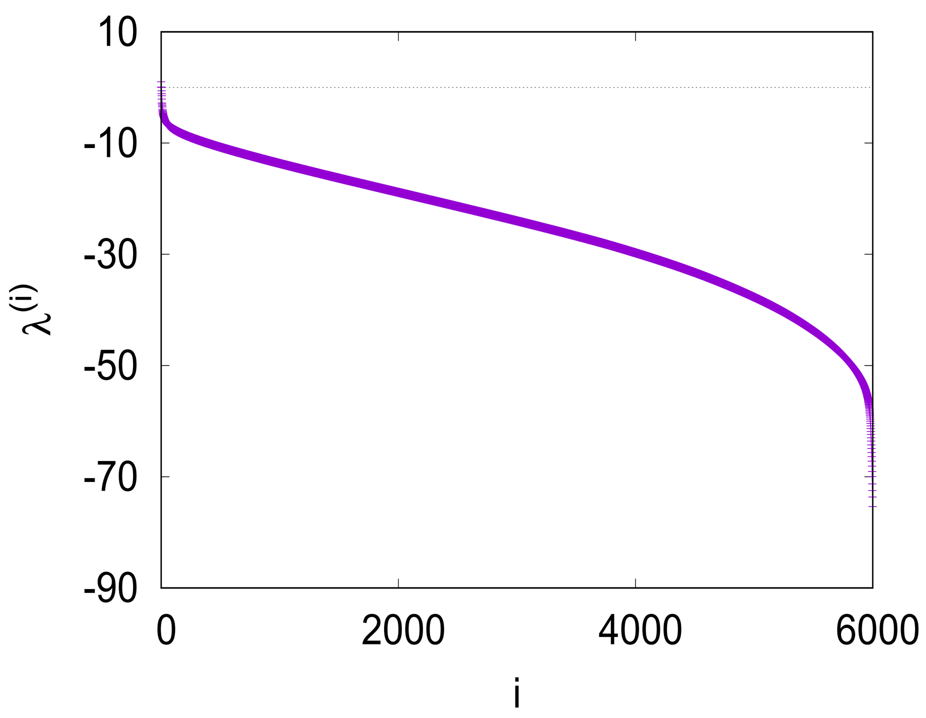

Figure 4: Lyapunov spectrum of the Lorenz-96 system.

Reference

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