Machine Learning Modeling of Dynamical Systems by using Biased Training Data

Yoshitaka Saiki (Hitotsubashi University)

1 Introduction.

Reservoir computing, a brain-inspired machine-learning technique that employs a data-driven dynamical system, is effective in predicting time series and frequency spectra in chaotic behaviors, including fluid flow and global atmospheric dynamics [1, 2, 4, 5, 6].

The extent to which a data-driven model using reservoir computing can capture the dynamical properties of original systems should be determined. Lu *et al.* [1] reported that a data-driven model has an attractor similar to that of the original system under an appropriate choice of parameters. Nakai and Saiki [3] confirmed that a single data-driven model could infer the time series of chaotic fluid flow from various initial conditions. Zhu *et al.* [7] identified some unstable periodic orbits of a data-driven model through delayed feedback control. They suggested that a data-driven model could reconstruct the attractor of the original dynamical system.

This study clarifies that a data-driven model using reservoir computing has richer information than that obtained from a training data, especially from dynamical system point of view, suggesting that dynamical properties of the original unknown dynamical system can be estimated by reservoir computing from a relatively short time series. The dynamical properties, such as Lyapunov exponents and manifold structures between stable and unstable manifolds, can be reconstructed by the data-driven model through reservoir computing.

2 Reservoir computation.

What's Reservoir computation?

- a relatively high-dimentional fixed neural-network composed of simple nonlinear dynamical systems
- determination of output layer
- For Lorenz system and Kuramoto–Sivashinsky system, inference [1, 5, 6]

2.1 Procedure of training.

◯ **1st step (making a reservoir vector)**
Making a reservoir vector $\mathbf{r}(t)$ correspond to decomposition of the input data $\mathbf{u}$ by using non-linear function tanh:
$\mathbf{r}(t+\Delta t) = (1-\alpha)\mathbf{r}(t) + \alpha \tanh(\mathbf{A}\mathbf{r}(t) + \mathbf{W}_{\rm in}\mathbf{u}(t))$.
$\mathbf{A}, \mathbf{W}_{\rm in}$: sparse random matrix, whose maximal eigenvalue is controlled.

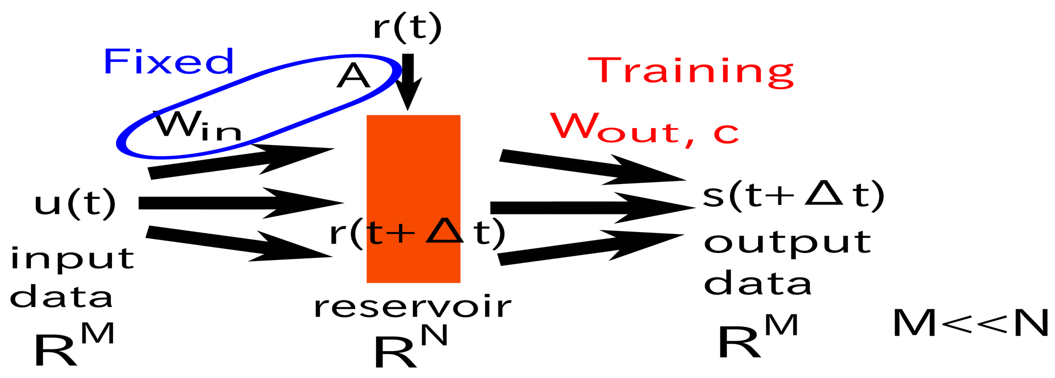

Figure 1: Schematic picture of a reservoir computing (training phase)

◯ **2nd step (determination of output layer)**
We determine $\mathbf{W}_{\rm out}$ s.t.

$$t^{\forall} < T \qquad \mathbf{W}_{\rm out}\mathbf{r}(t+\Delta t) \approx \mathbf{s}(t+\Delta t).$$

⇒ Determine them s.t. the form is minimized:

$$\sum_{l=1}^{L} \|(\mathbf{W}_{\rm out}\mathbf{r}(l\Delta t)) - \mathbf{s}(l\Delta t)\|^2 + \beta[Tr(\mathbf{W}_{\rm out}\mathbf{W}_{\rm out}^T)].$$

2.2 Procedure of inference.

Using the $\mathbf{W}^*_{\rm out}$, we infer the time-series $\mathbf{s}$.

$\hat{\mathbf{s}}(t) = \mathbf{W}^*_{\rm out}\mathbf{r}(t)$
$\mathbf{r}(t+\Delta t) = (1-\alpha)\mathbf{r}(t) + \alpha \tanh(\mathbf{A}\mathbf{r}(t) + \mathbf{W}_{\rm in}\hat{\mathbf{s}}(t))$.

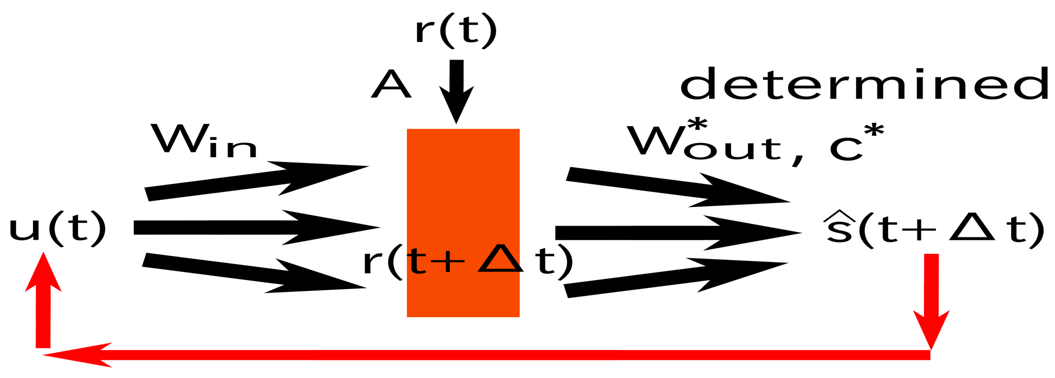

Figure 2: Schematic picture of a reservoir computing (prediction phase)

This reservoir system corresponds to the data-driven model of u.

3 Our project.

In our project by employing a reservoir computing we evaluate a dynamical model constructed from a biased training data. It is well known that an infinitely many periodic orbits are embedded in a chaotic attractor. We create training data by choosing a set of periodic orbits. In our preliminary study we have confirmed that a model trained from a set of relatively low periods periodic orbits reconstructs a trajectory which approximates the actual one for a certain amount of time. For the Lorenz system with classical parameter values, each periodic orbit is coded by the symbol sequence L or R depending on the sign of the x coordinate along a periodic orbit. A typical long trajectory has similar number of Ls and Rs. Therefore, a periodic orbit with a symbol sequence such as $L^{10}R^1$ (the number of Ls is much bigger than that of Rs) can be used as a biased training data. In this study we evaluate each of the models trained from different biased training data.

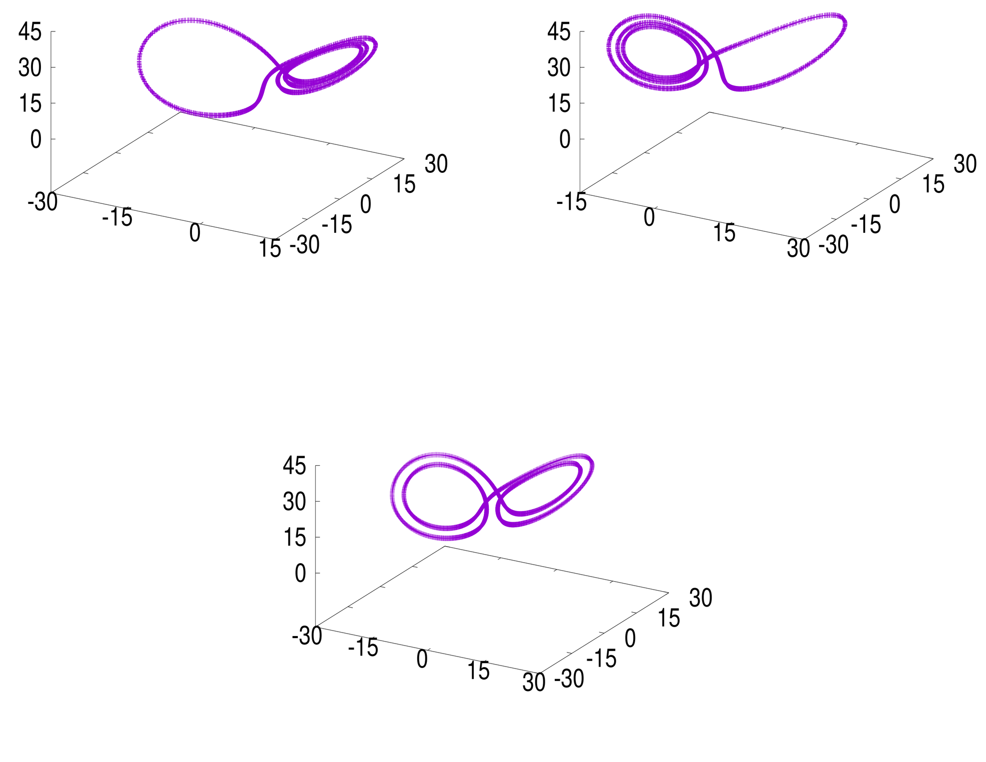

Figure 3: Examples of three periodic trajectories used as training data

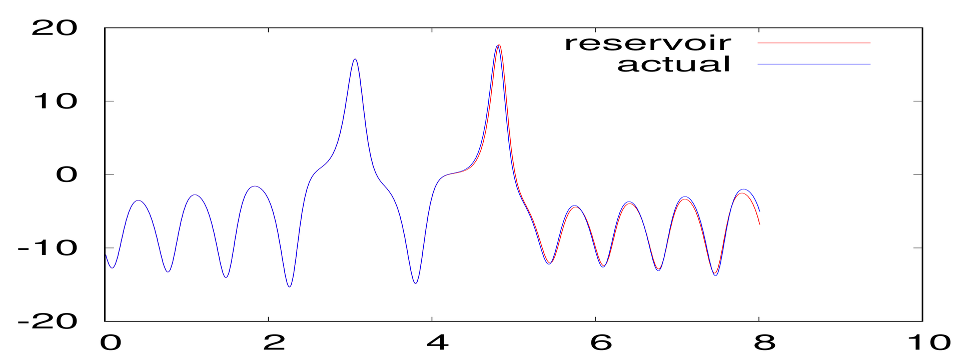

Figure 4: Time development of x of a model trajectory in compared to that of an actual trajectory

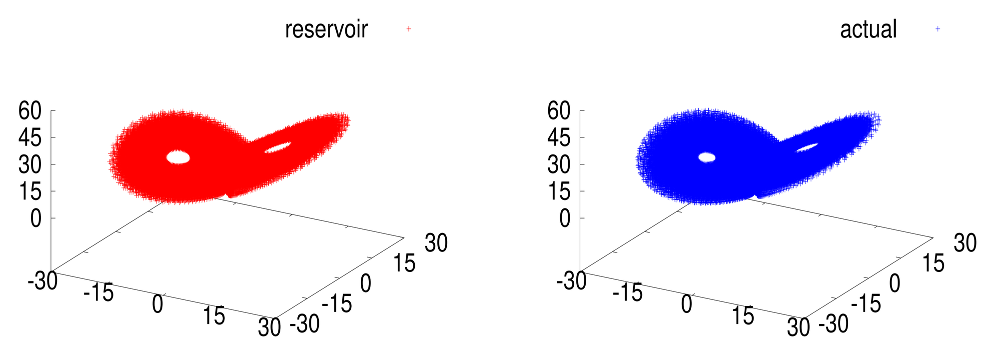

Figure 5: Projection of a long model trajectory and that of a long actual trajectory

Reference

[1] Z. Lu, B. R. Hunt, and E. Ott, Chaos 28, 061104 (2018).
[2] K. Nakai and Y. Saiki, Physical Review E 98, 023111 (2018).
[3] K. Nakai and Y. Saiki, Discrete Contin. Dyn. Syst. S 14, 3641079 (2021).
[4] M. U. Kobayashi, K. Nakai, Y. Saiki and N. Tsutsumi, Physical Review E 104 (4), 044215 (2021).
[5] J. Pathak, Z. Lu, B. Hunt, M. Girvan, and E. Ott, Chaos 27, 121102 (2017).
[6] J. Pathak, B. Hunt, M. Girvan, Z. Lu, and E. Ott, Phys. Rev. Lett. 120, 024102 (2018).
[7] Q. Zhu, H. Ma, and W. Lin, Chaos 29, 093125 (2019).