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研究課題名　機械学習に基づく流体変数の未来予測と数学的背景

1 Introduction

Most fluid flow is governed by 3 dimensional incompressible Navier–Stokes equation.
⇒It is hard to derive an equation describing macroscopic dynamics analytically from NS.

Aim of investigation

Learning limited time-series data of fluid flow
⇒ Construction of a data-driven model, especially for macroscopic variables.

⇒ Using Machine-learning technique, we deal with energy functions:

$$E_0(k,t) := \frac{1}{2}\int_{D_k} |\mathcal{F}_{[v]}(K,t)|^2 dK,$$

where $D_k := \{K \in \mathbb{R}^3 \mid k - 0.5 \le |K| < k + 0.5\}$. To get rid of the high-frequency fluctuation, we take short-time average $E(k,t) = \sum_{s=t-99\Delta t}^{t} E_0(k,s)/100$. Then, $\tilde{E}(k,t)$ is the normalized value of each variable $E(k,t)$.

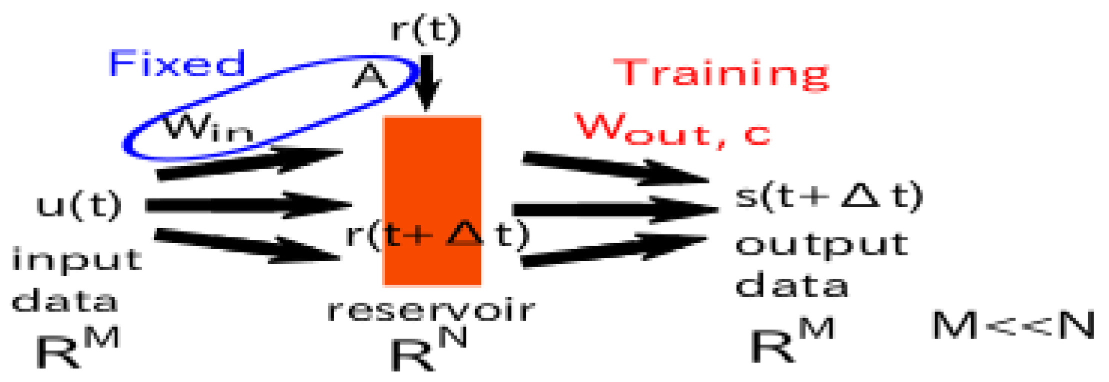
2 Reservoir computation

What's Reservoir computation?

- a relatively high-dimentional fixed neural-network composed of simple nonlinear dynamical systems
- determination of output layer
- For Lorenz system and Kuramoto–Sivashinsky system, inference (Lu et al. (2017), Pathak et al. (2017,2018)).

2.1 Procedure of training

○ **1st step (making a reservoir vector)**
Making a reservoir vector $\mathbf{r}(t)$ correspond to decomposition of the input data $\mathbf{u}$ by using nonlinear function tanh:
$\mathbf{r}(t+\Delta t) = (1-\alpha)\mathbf{r}(t) + \alpha \tanh(\mathbf{A}\mathbf{r}(t) + \mathbf{W}_{\rm in}\mathbf{u}(t))$.
$\mathbf{A}, \mathbf{W}_{\rm in}$: sparse random matrix, whose maximal eigenvalue is controlled.



○ **2nd step (determination of output layer)**
We determine the $\mathbf{W}_{\rm out}$ and c s.t.

$$t^{\forall} < T \qquad \mathbf{W}_{\rm out}\mathbf{r}(t+\Delta t) + c \approx \mathbf{s}(t+\Delta t).$$

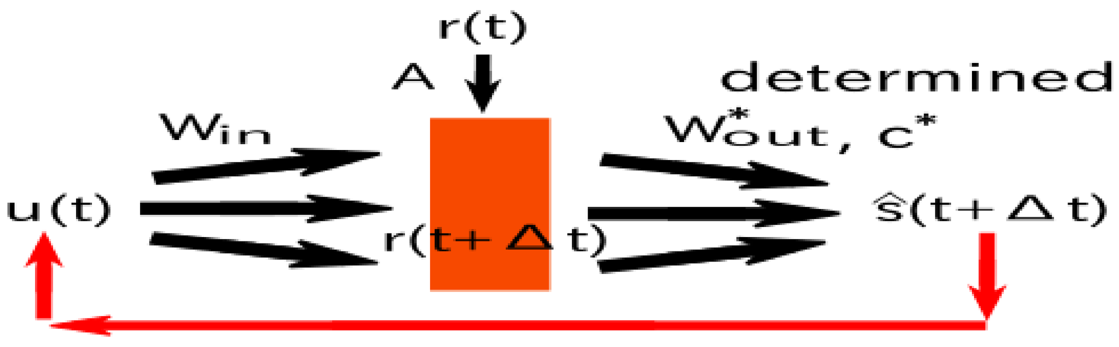
⇒ Determine them s.t. the form is minimized:

$$\sum_{l=1}^{L} \|(\mathbf{W}_{\rm out}\mathbf{r}(l\Delta t) + \mathbf{c}) - \mathbf{s}(l\Delta t)\|^2 + \beta[Tr(\mathbf{W}_{\rm out}\mathbf{W}_{\rm out}^T)].$$

2.2 Procedure of inference

Using the $\mathbf{W}^*_{\rm out}$ and c^*, we infer the time-series $\mathbf{s}$.

$\hat{\mathbf{s}}(t) = \mathbf{W}^*_{\rm out}\mathbf{r}(t) + c^*$
$\mathbf{r}(t+\Delta t) = (1-\alpha)\mathbf{r}(t) + \alpha \tanh(\mathbf{A}\mathbf{r}(t) + \mathbf{W}_{\rm in}\hat{\mathbf{s}}(t))$.



This reservoir system corresponds to the data-driven model of u.

3 Results

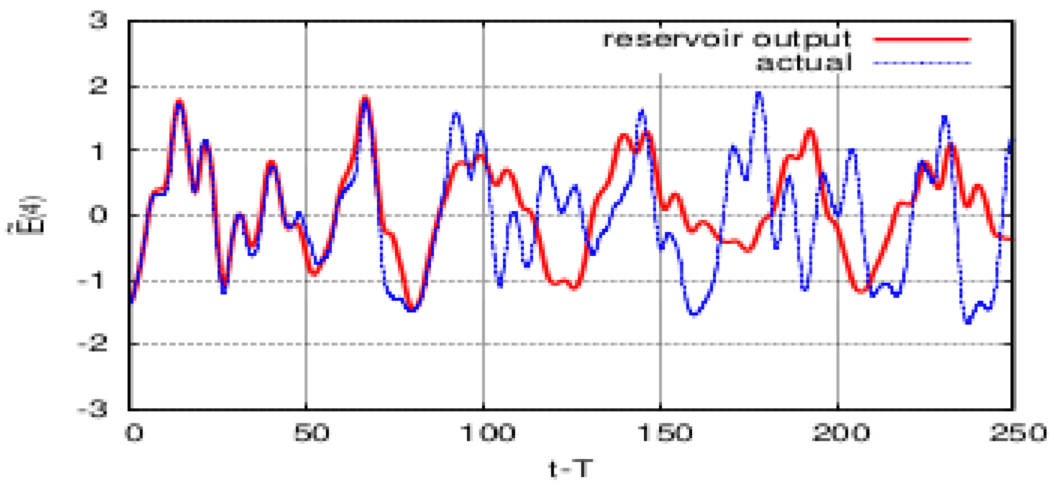
3.1 Inference of macroscopic variables

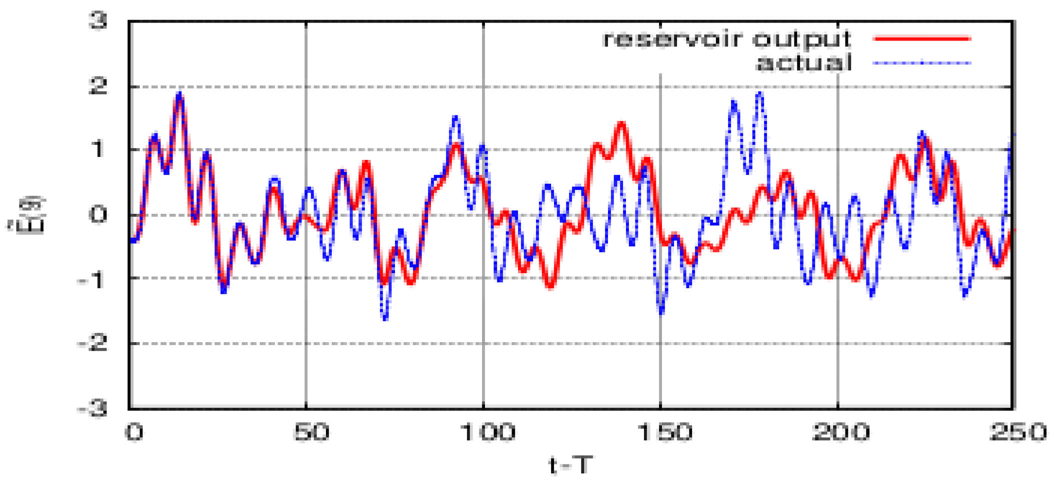
In the training phase for $t \in (0,T]$, $\mathbf{W}^*_{\rm out}$ and c^* are determined by setting

$$\mathbf{u}(t) = (\tilde{E}(1,t), \tilde{E}(2,t), \cdots, \tilde{E}(9,t))^t,$$
$$\mathbf{s}(t) = (\tilde{E}(1,t), \tilde{E}(2,t), \cdots, \tilde{E}(9,t))^t.$$

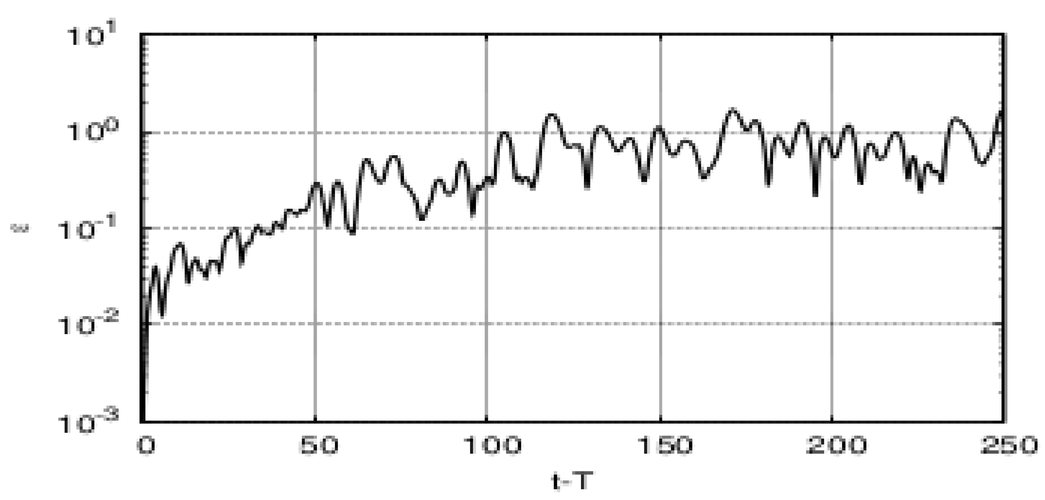
In short, we make a model inferring $E(k)$ using $E(k)$ at the previous step.

○ **Inference of time-series of $E(k)$.**
When $t - T < 100$, inferred time-series data obtained from our reservoir system (red line) almost coincides with that of reference data obtained from the DNS of NS (blue line).

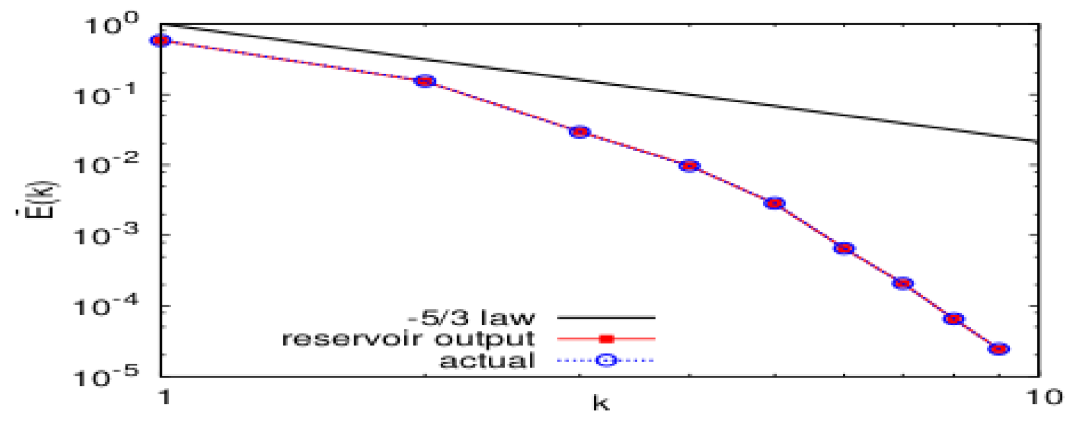




○ **Error of inference**
The inference error defined by $\varepsilon(t) = \sum_{k=1}^{N_0} |\tilde{E}(k,t) - \hat{\tilde{E}}(k,t)|/N_0$ ($N_0 = 9$) is shown to grow exponentially with time up to $t - T = 100$, which is inevitable for a chaotic behavior of a fluid flow.



○ **Energy spectrum from inference data**
A time average of an energy function $\overline{E}(k)$ obtained from the inference procedure of $E(k,t)$ for $t - T > 1000$ (after the time-series inference fails) coincides with that from a reference data obtained from the direct numerical simulation of NS.

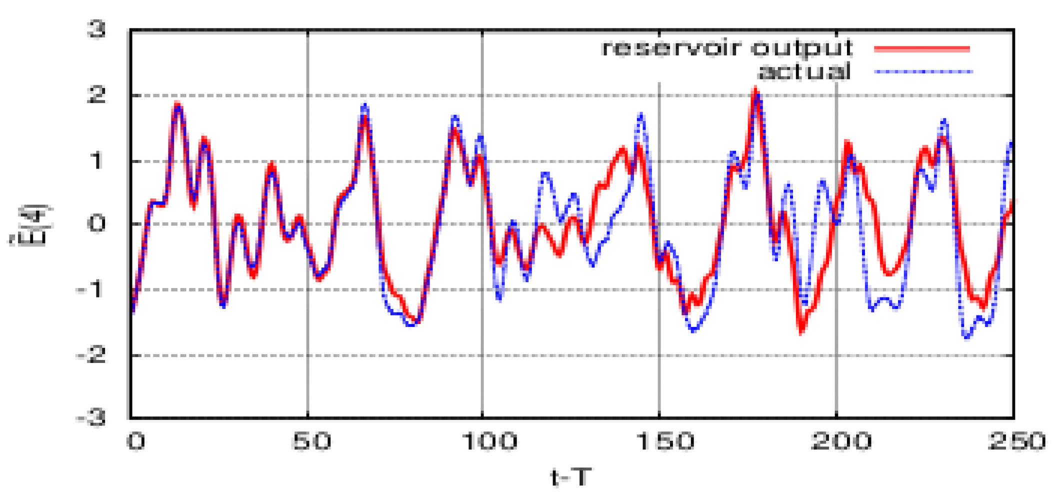


3.2 Inference of macroscopic variable from only one measurement using delay coordinates

The number of measurements is limited
⇒ Delay coordinates are helpful.
Under an assumption that the accessible measurement is limited to only $E(4,t)$, $\mathbf{W}^*_{\rm out}$ and c^* are determined by using delay coordinates:

$$\mathbf{u}(t) = (\tilde{E}(4,t), \tilde{E}(4,t-\Delta\tau), \cdots, \tilde{E}(4,t-35\Delta\tau))^t,$$
$$\mathbf{s}(t) = (\tilde{E}(4,t), \tilde{E}(4,t-\Delta\tau), \cdots, \tilde{E}(4,t-35\Delta\tau))^t,$$

where $\Delta\tau = 2.5$. **In short, we make a model inferring $E(4)$ using only $E(4)$ with delay coordinates at the previous step.**



The inference of $\tilde{E}(4,t)$ is as successful as the case when there are 9 measurements.

4 Conclusion

- We have succeeded in inferring time-series of macroscopic variables of a fluid flow.
- The average of the inferred time-series of macroscopic variables coincides with that from a reference data obtained from the direct numerical simulation of the NS.
- When the number of measurements is limited, and less than the Lyapunov dimensions, delay coordinates are useful.

Conclusion

We have succeeded in constructing a closed form equation of macroscopic behavior of a fluid flow only from macroscopic data without prior knowledge of physical process.

Reference

[1] K. Nakai and Y. Saiki, Physical Review E 98, 023111 (2018).